

THE DISK COMPLEX AND 2-BRIDGE KNOTS

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ABSTRACT. We give an alternative proof of a result of Kobayashi and Saeki that every genus one 1-bridge position of a non-trivial 2-bridge knot is a stabilization.

1. INTRODUCTION

A *genus one 1-bridge position* of a knot K in S^3 , simply a $(1, 1)$ -position of K , is a splitting of S^3 into two solid tori such that K intersects each of the solid tori in a properly embedded trivial arc. Similarly, a *2-bridge position* of a knot K in S^3 is a splitting of S^3 into two 3-balls such that K intersects each of the 3-balls in a pair of properly embedded trivial arcs. Here, a properly embedded arc α is said to be *trivial* if there exists an arc β in the boundary such that $\alpha \cup \beta$ forms a loop that bounds a disk. A knot which admits a 2-bridge position is called a *2-bridge knot*. Let α be one of the four arcs in a 2-bridge position of K , that is, α is a trivial arc in one of the two 3-balls, say B . Let $N(\alpha)$ be a regular neighborhood of α in B . Then the splitting of S^3 into the two solid tori $\overline{B - N(\alpha)}$ and $\overline{S^3 - B \cup N(\alpha)}$ turns out to be a $(1, 1)$ -position of K , which we call a *stabilization* of the 2-bridge position.

It is known that the 2-bridge position of a 2-bridge knot K is unique up to equivalence, proved by H. Schubert [7], so there are exactly four equivalence classes of stabilization depending on the choice of one of the four arcs. Further T. Kobayashi and O. Saeki showed that any $(1, 1)$ -position of a 2-bridge knot can be obtained in this way, which is the main result of [6] stated as follows.

Theorem 1.1. *Every $(1, 1)$ -position of a non-trivial 2-bridge knot is a stabilization.*

In this work, we give an alternative proof of Theorem 1.1. The key idea is to consider the 2-fold cover L of S^3 branched along a 2-bridge knot K and the covering involution ϕ of L over S^3 . It is well known that L is a lens space and the preimages of the two solid tori of a $(1, 1)$ -position of K are genus two handlebodies, say V and W . Each is invariant under the involution ϕ , as is the preimage of K . We will construct a simplicial complex $\mathcal{PT}(V)$ for the handlebody V , called the *primitive tree*. Then ϕ defines a simplicial involution of $\mathcal{PT}(V)$ and the existence of a fixed point of the involution on $\mathcal{PT}(V)$ enables us to find some special meridian disks on the two solid tori of the $(1, 1)$ -position.

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Consequently, we find a 2-bridge position of K which stabilizes to the original $(1, 1)$ -position.

In Section 2, we introduce a well-known simplicial complex, called the *non-separating disk complex* for a genus-2 handlebody V . When V is one of the handlebodies of a genus-2 Heegaard splitting for a lens space, the non-separating disk complex for V admits a special subcomplex, called the *primitive disk complex*. In Section 3, the combinatorial structure of the primitive disk complex for each lens space is described, which was done in the previous works [3], [4] and [5]. From the primitive disk complex, we construct the *primitive tree* and introduce some properties of the simplicial automorphisms of the complex that we need. The proof of Theorem 1.1 will be given in the final section.

We use the standard notation $L = L(p, q)$ for a lens space, where its first homology group $H_1(L)$ is the cyclic group of order $|p|$. We refer [9] to the reader. The integer p can be assumed to be positive, and any two lens spaces $L(p, q)$ and $L(p', q')$ are diffeomorphic if and only if $p = p'$ and $q'q^{\pm 1} \equiv \pm 1 \pmod{p}$. Thus, we will assume $1 \leq q \leq p/2$. For convenience, we will not distinguish a disk (or union of disks) and a diffeomorphism from their isotopy classes in their notation. We will denote by $N(X)$ and \bar{X} a regular neighborhood of X and the closure of X for a subspace X of a space, where the ambient space will be clear from the context.

Finally we remark that the key idea of this work came from Darryl McCullough. The authors are deeply grateful to him for his valuable advice and comments.

2. THE NON-SEPARATING DISK COMPLEX OF A GENUS-2 HANDLEBODY

The *non-separating disk complex* for a genus-2 handlebody V , denoted by $\mathcal{D}(V)$, is a simplicial complex whose vertices are the isotopy classes of essential non-separating disks in V such that a collection of $k + 1$ vertices spans a k -simplex if and only if it admits a collection of representative disks which are pairwise disjoint. It is easy to see that $\mathcal{D}(V)$ is 2-dimensional and every edge of $\mathcal{D}(V)$ is contained in infinitely but countably many 2-simplices. In [8], it is proved that $\mathcal{D}(V)$ and the link of any vertex of $\mathcal{D}(V)$ are all contractible. Thus, the dual complex of $\mathcal{D}(V)$ is a tree, which we call the *dual tree* of $\mathcal{D}(V)$ simply. The dual tree is the subcomplex of the barycentric subdivision of $\mathcal{D}(V)$ spanned by the barycenters of the 1-simplices and 2-simplices. See Figure 1.

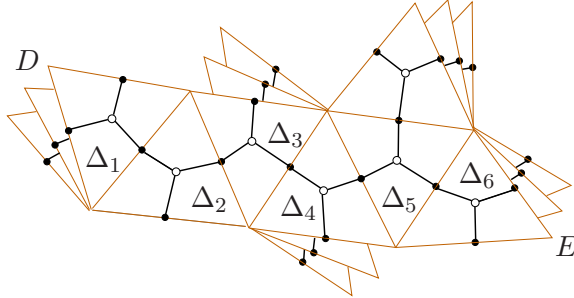


FIGURE 1. A portion of the disk complex $\mathcal{D}(V)$ and the dual complex, and the corridor connecting D and E .

Let D and E be vertices of $\mathcal{D}(V)$ that are not adjacent to each other. In the barycentric subdivision of $\mathcal{D}(V)$, the links of the vertices D and E are disjoint trees, and hence there

exists a unique shortest path in the dual tree of $\mathcal{D}(V)$ connecting the two links. Let $v_1, w_1, v_2, w_2, \dots, v_{n-1}, w_{n-1}, v_n$ be the sequence of the consecutive vertices of this path. We note that v_i is the barycenter of a 2-simplex, denoted by Δ_i , for each $i \in \{1, 2, \dots, n\}$. The full subcomplex of $\mathcal{D}(V)$ spanned by the vertices of $\Delta_1, \Delta_2, \dots, \Delta_n$ is called the *corridor* connecting D and E , and we denote it just by the sequence $\{\Delta_1, \Delta_2, \dots, \Delta_n\}$. The vertices D and E are the vertices of Δ_1 and Δ_n respectively, which are not contained in the edges $\Delta_1 \cap \Delta_2$ and $\Delta_{n-1} \cap \Delta_n$ respectively.

3. THE PRIMITIVE TREES

We denote by $(V, W; \Sigma)$ a genus-2 Heegaard splitting of a lens space $L = L(p, q)$. That is, V and W are genus-2 handlebodies such that $V \cup W = L$ and $V \cap W = \partial V = \partial W = \Sigma$, a genus-2 closed orientable surface in L . A disk E properly embedded in V is said to be *primitive* if there exists a disk E' properly embedded in W such that the two loops ∂E and $\partial E'$ intersect transversely in a single point. We call such a disk E' a *dual disk* of E , which is also primitive in W having a dual disk E . The pair of any two disjoint, non-isotopic primitive disks D and E in V is called simply a *primitive pair*, and if a disk E' is a dual disk of both D and E , then E' is called a *common dual disk* of the pair. Primitive disks are necessarily non-separating, and both $W \cup N(E)$ and $V \cup N(E')$ are solid tori.

The *primitive disk complex* $\mathcal{P}(V)$ for the splitting is then defined to be the full subcomplex of $\mathcal{D}(V)$ spanned by the vertices of primitive disks. For each primitive disk E in V , it is easy to find infinitely many non-isotopic primitive disks in V disjoint from E , so each vertex of $\mathcal{P}(V)$ has infinite valency for any lens space. The primitive disk complex $\mathcal{P}(W)$ for W is defined in the same way, which is isomorphic to $\mathcal{P}(V)$, since it is known that any two genus-2 Heegaard splittings of a lens space are isomorphic to each other (see [1] and [2]).

The combinatorial structure of $\mathcal{P}(V)$ for each lens space was fully studied in [3], [4] and [5]. We describe it as follows. To make the statement simple, we will say that an edge of $\mathcal{P}(V)$ is of *type-0* (*type-1*, *type-2*, respectively) if, up to isotopy, a primitive pair representing the end vertices of the edge has no common dual disk (has a unique common dual disk, has exactly two common dual disks which form a primitive pair in W , respectively).

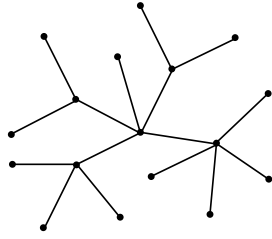
Lemma 3.1 (Theorem 4.5 [4]). *Let $L = L(p, q)$ be a lens space with $1 \leq q \leq p/2$, and let $(V, W; \Sigma)$ be a genus-2 Heegaard splitting of L . If $p \equiv \pm 1 \pmod{q}$, then the primitive disk complex $\mathcal{P}(V)$ is contractible, and we have one of the following cases.*

- (1) *If $q \neq 2$ and $p \neq 2q+1$, then $\mathcal{P}(V)$ is a tree, and every vertex has infinite valency. In this case,*
 - (a) *if $p = 2$ and $q = 1$, then every edge is of type-2.*
 - (b) *if $p \geq 4$ and $q = 1$, then every edge is of type-1.*
 - (c) *if $q \neq 1$, then every edge is of either type-0 or type-1, and infinitely many edges of type-0 and of type-1 meet in each vertex.*
- (2) *If $q = 2$ or $p = 2q + 1$, then $\mathcal{P}(V)$ is 2-dimensional, and every vertex meets infinitely many 2-simplices. In this case,*
 - (a) *if $p = 3$, then every edge is of type-1 and is contained in a unique 2-simplex.*

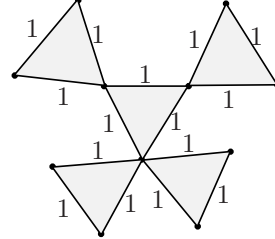
- (b) if $p = 5$, then every edge is of either type-0 or type-1. Among the three edges of a 2-simplex, exactly one is of type-1 and the remaining two are of type-0. Every edge of type-0 is contained in exactly two 2-simplices, while every edge of type-1 is in a unique 2-simplex.
- (c) if $p \geq 7$, then every edge is of either type-0 or type-1. Among the three edges of a 2-simplex, exactly one is of type-1 and the remaining two are of type-0. Every edge of type-0 is contained in a unique 2-simplex. Every edge of type-1 is contained in a unique 2-simplex or in no 2-simplex.

If $p \not\equiv \pm 1 \pmod{q}$, then $\mathcal{P}(V)$ is not connected, and it consists of infinitely many tree components. All the tree components are isomorphic to each other. Any vertex of $\mathcal{P}(V)$ has infinite valency, and further, infinitely many edges of type-0 and of type-1 meet in each vertex.

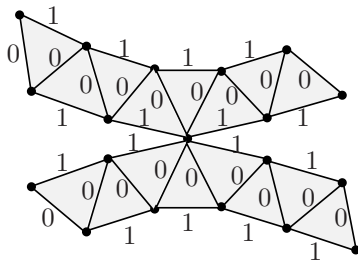
Figure 2 illustrates a small portion of the primitive disk complex $P(V)$ for each case. The numbers 0 or 1 in the figure indicate the type of the edges.



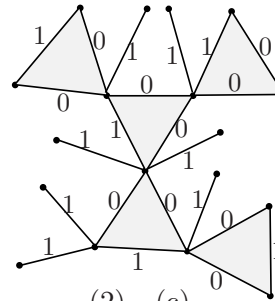
(1) - (a), (b), (c)



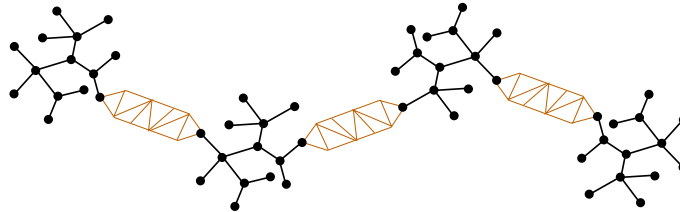
(2) - (a)



(2) - (b)



(2) - (c)



(3) A portion of $\mathcal{P}(V)$ with bridges for the case of $p \not\equiv \pm 1 \pmod{q}$

FIGURE 2. A portion of primitive disk complex $\mathcal{P}(V)$ for each lens space.

In the case of $p \not\equiv \pm 1 \pmod{q}$, that is, $\mathcal{P}(V)$ consists of infinitely many tree components, we can define a special kind of corridor in $\mathcal{D}(V)$, call a *bridge*. A bridge is a corridor connecting the vertices D and E of two primitive disks and contains no vertices of primitive disks except D and E . From the structure of $\mathcal{D}(V)$ (the dual complex of $\mathcal{D}(V)$ is a tree), any bridge connects exactly two tree components of $\mathcal{P}(V)$, and any two tree components of $\mathcal{P}(V)$ is connected by at most a single bridge. Note that the union of $\mathcal{P}(V)$ and the bridges form a contractible subcomplex of $\mathcal{D}(V)$. Figure 2 (3) describes a small portion of four tree components of $\mathcal{P}(V)$ and three bridges connecting them consecutively. Further, it was shown in [5] that any two bridges are isomorphic to each other, and any two bridges are either disjoint from each other or intersect only in an end vertex (Lemma 2.14 in [5]).

The primitive disk complex $\mathcal{P}(V)$ for a genus-2 Heegaard splitting $(V, W; \Sigma)$ of a lens space was used to obtain a finite presentation of the *genus-2 Goeritz group* of a lens space, denoted by \mathcal{G} . Here the genus-2 Goeritz group \mathcal{G} is the mapping class group of the splitting, that is, the group of the isotopy classes of orientation preserving diffeomorphisms of the lens space preserving the two handlebodies V and W setwise. The group \mathcal{G} acts on the complex $\mathcal{P}(V)$ simplicially, and the action was fully studied in [3], [4] and [5], and a finite presentation of \mathcal{G} of each lens space was obtained. The followings are somewhat technical results on the action of \mathcal{G} on $\mathcal{P}(V)$ already developed in the previous works, which we will need in the next section.

Lemma 3.2. *Let $L = L(p, q)$ be a lens space with $1 \leq q \leq p/2$, and let $(V, W; \Sigma)$ be a genus-2 Heegaard splitting of L . Let ϕ be an order-2 element of the Goeritz group \mathcal{G} , and let $\bar{\phi}$ be the simplicial automorphism of $\mathcal{P}(V)$ and of $\mathcal{P}(W)$ defined by ϕ .*

- (1) *If $\bar{\phi}$ exchanges the end vertices of an edge of type-2 of $\mathcal{P}(V)$, then $\bar{\phi}$ preserves each of the two vertices of $\mathcal{P}(W)$ of the two common dual disks of the primitive pair representing the end vertices.*
- (2) *If $\bar{\phi}$ exchanges the end vertices of an edge of type-0 or a bridge, then ϕ induces the isomorphism ϕ_* of the first homology group $H_1(L)$ such that $\phi_*(x) = qx$.*

We remark that we have the edge of type-2 only when $p = 2$, the edge of type-0 only when $q > 2$, and the bridge only when $p \not\equiv \pm 1 \pmod{q}$.

Proof. (1) This follows directly from Lemma 5.2 (1) in [3], which states that the subgroup $\mathcal{G}_{\{D \cup E\}}$ of \mathcal{G} preserving the union of the end vertices D and E of an edge of type-2 is the dihedral group $\langle \rho, \gamma \mid \rho^4 = \gamma^2 = (\rho\gamma)^2 = 1 \rangle$ of order 8. The element ρ and γ are described in Figure 8 in [3]. In the figure, γ preserves each of the vertices D and E but exchanges the two vertices D' and E' of the two common dual disks of the pair $\{D, E\}$, while ρ exchanges both of D, E and D', E' . Thus, an order-2 element $\bar{\phi}$ exchanging D and E is either $\rho\gamma$ or $\rho^3\gamma$, and we see that both are preserving each of D' and E' .

(2) The key argument was already provided in the proofs of Lemma 3.13 in [4] (for the edge of type-0) and Lemma 2.13 in [5] (for the bridge). We sketch the argument briefly. We first have that there exists an element of \mathcal{G} exchanging the end vertices of an edge of type-0 or a bridge if and only if $q^2 \equiv 1 \pmod{p}$, from Lemma 5.3 (3) in [4] (for the edge of type-0) and from Lemma 4.7 (2) in [5] (for the bridge). (In the case of the bridge, for

any two tree components \mathcal{T}_1 and \mathcal{T}_2 connected by a bridge, it was shown that there is no element of \mathcal{G} exchanging \mathcal{T}_1 and \mathcal{T}_2 if $q^2 \not\equiv 1 \pmod{p}$.) Thus, with $q^2 \equiv 1 \pmod{p}$, we can replace q' by q , and \bar{q} by q in the proofs of Lemma 3.13 in [4] and Lemma 2.13 in [5] respectively, where q' is the unique integer satisfying $qq' \equiv \pm 1 \pmod{p}$ and \bar{q} is one of q and q' .

Let D and E be the primitive disks in V representing the vertices of the edge of type-0 or a bridge of $\mathcal{P}(V)$. Then $V_D = \overline{V - N(D)}$ and $V_E = \overline{V - N(E)}$ are solid tori, that form genus-1 Heegaard splittings of L with their exteriors $W_D = \overline{L - V_D}$ and $W_E = \overline{L - V_E}$ respectively. In the proof of Lemma 3.13 in [4], the core circles of V_D and V_E , denoted by l_D and l_E , represent the generators $[l_D]$ and $[l_E]$ of $H_1(V_D)$ and $H_1(V_E)$ respectively. In the proof of Lemma 2.13 in [5], the core circles of W_D and W_E , denoted by l_D and l_E again, represent the generators $[l_D]$ and $[l_E]$ of $H_1(W_D)$ and $H_1(W_E)$ respectively. In either cases, it was shown that $[l_D]$ and $[l_E]$ satisfy $[l_D] = q[l_E]$ in $H_1(L)$ after a suitable choice of orientation of l_D and l_E . The elements ϕ sends E to D and hence the isotopy class of l_E to that of l_D . That is, ϕ induces ϕ_* such that $\phi_*(1) = q$. \square

Now we are ready to construct the primitive tree for each lens space. Given a lens space $L(p, q)$, $1 \leq q \leq p/2$, with a genus-2 Heegaard splitting $(V, W; \Sigma)$, the *primitive tree* $\mathcal{PT}(V)$ is defined as follows.

- (1) If $p \equiv \pm 1 \pmod{q}$, and $q \neq 2$ and $p \neq 2q + 1$, then $\mathcal{P}(V)$ is already a tree. So we just define $\mathcal{PT}(V)$ to be $\mathcal{P}(V)$. We observe that there are essentially three different kind of trees from Lemma 3.1 (1):
 - (a) every edge is of type-2 if $p = 2$,
 - (b) every edge is of type-1 if $q = 1$, and
 - (c) every edge is of either type-0 or type-1 otherwise.
- (2) If $p \equiv \pm 1 \pmod{q}$, and $q = 2$ or $p = 2q + 1$, then $\mathcal{P}(V)$ is 2-dimensional. We have the three cases as stated in Lemma 3.1 (2).
 - (a) If $p = 3$, then the primitive disk complex $\mathcal{P}(V)$ deformation retracts to a tree as shown in Figure 3 (2)-(a). We define $\mathcal{PT}(V)$ to be the resulting tree. In the figure, the black vertices are the vertices of $\mathcal{P}(V)$ while the white ones are the barycenters of 2-simplexes of $\mathcal{P}(V)$.
 - (b) If $p = 5$, then we remove every type-1 edge of $\mathcal{P}(V)$ to get a tree. That is, $\mathcal{PT}(V)$ is the subcomplex of $\mathcal{P}(V)$ containing only the edges of type-0. See Figure 3 (2)-(b).
 - (c) If $p \geq 7$, then we remove every type-1 edge “contained in a 2-simplex” of $\mathcal{P}(V)$ to get a tree. Then $\mathcal{PT}(V)$ is the subcomplex of $\mathcal{P}(V)$ containing only the edges of type-0, and the edges of type-1 that was not contained in any 2-simplex of $\mathcal{P}(V)$. See Figure 3 (2)-(c).
- (3) If $p \not\equiv \pm 1 \pmod{q}$, the primitive disk complex $\mathcal{P}(V)$ consists of infinitely many tree components. If there exists a bridge connecting the vertices D and E of $\mathcal{P}(V)$, we replace the bridge by a new edge connecting D and E . Then we define $\mathcal{PT}(V)$ to be the union of $\mathcal{P}(V)$ and all the new edges came from the bridges. We call such a new edge just a *bridge* again. See Figure 3 (3).

We remark that every vertex of the primitive tree $\mathcal{PT}(V)$ is that of the primitive complex $\mathcal{P}(V)$, the isotopy class of a primitive disk in V except the case of $L(3, 1)$. For

$L(3,1)$, the tree is bipartite and we have two kind of vertices; the black, the vertex of $\mathcal{P}(V)$, has infinite valency while the white, the barycenter of a 2-simplex of $\mathcal{P}(V)$, has valency three.

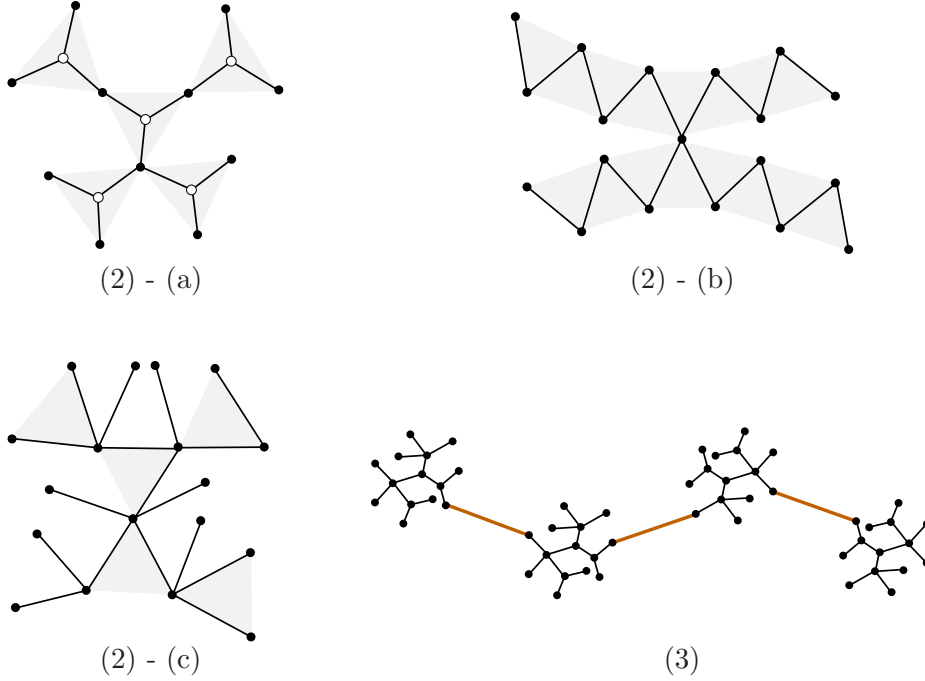


FIGURE 3. A portion of primitive tree $\mathcal{PT}(V)$ for each lens space in the cases (2) and (3).

4. PROOF OF THE MAIN THEOREM

In this section, we prove Theorem 1.1 using our previous results. Throughout the section, all the symbols for disks and diffeomorphisms denotes only themselves, not their isotopy classes. Let K be a knot in $(1,1)$ -position with respect to the Heegaard splitting $(V_0, W_0; \Sigma_0)$ of S^3 . That is, each of V_0 and W_0 is a solid torus that K intersects in a properly embedded trivial arc. Let L be the 2-fold cover of S^3 branched along K . It is well-known that L is a lens space $L(p, q)$ and the preimages of V_0 and W_0 are genus-2 handlebodies V and W that form a genus-2 Heegaard splitting of L . Each of V and W is invariant under the covering involution ϕ of L over S^3 , as is the preimage \hat{K} of K . Thus the isotopy class of ϕ is an element of the genus-2 Goeritz group \mathcal{G} of L . We note that ϕ induces a homomorphism ϕ_* on the first homology group $H_1(L)$ as $\phi_*(x) = -x$.

Lemma 4.1. *Let V and W be the genus-2 handlebodies and let ϕ be the involution defined as in the above. Let D be a disk properly embedded in V or in W , say in V , such that $\partial D = \partial\phi(D)$ or ∂D intersects $\partial\phi(D)$ transversely. Let ℓ be a simple closed curve on ∂V such that $\phi(\ell) = \ell$ and ℓ intersects ∂D transversely in a single point.*

- (1) *There exists a disk E properly embedded in V such that $\partial E = \partial D$ and $E \cap \phi(E)$ contains no loops in the interior.*
- (2) *There exists a disk F properly embedded in V such that F is disjoint from $\phi(F)$, and ∂F intersects ℓ in a single point.*

Proof. (1) Suppose that $D \cap \phi(D)$ contains loops in the interior. Let γ be an innermost loop of $D \cap \phi(D)$ on $\phi(D)$. That is, γ cuts off a subdisk D_1 from $\phi(D)$ such that $D_1 \cap D = \gamma$. The loop γ also bounds a disk D_0 on D . Replacing D with $\overline{D \setminus D_0} \cup D_1$ and repositioning by a small isotopy produces a new disk D_* such that $\partial D_* = \partial D$ and $D_* \cap \phi(D_*)$ contains fewer loops than $D \cap \phi(D)$ had. Repeating the process finitely many times, we finally get a disk E satisfying the required condition.

(2) Suppose that ℓ intersects ∂D in a single point, say z . Then ℓ intersects $\partial \phi(D)$ only in $\phi(z)$. By (1), we may assume that $D \cap \phi(D)$ contains no loops in the interior. Let β be an outermost arc of $D \cap \phi(D)$ on $\phi(D)$. That is, β cuts off a subdisk D'_1 from $\phi(D)$ such that $D'_1 \cap D = \beta$. Further, we may choose β so that ∂D_1 does not contain the point $\phi(z)$. The arc β also cuts off a disk D'_0 from D that does not contain z . Replacing D with $\overline{D \setminus D'_0} \cup D'_1$ and repositioning by a small isotopy produces a new dual disk D'_* such that $D'_* \cap \phi(D'_*)$ has fewer arcs than $D \cap \phi(D)$ had. Repeating the process finitely many times, we finally get a disk F satisfying the required condition. \square

Lemma 4.2. *Let V and W be the genus-2 handlebodies and let ϕ be the involution defined as in the above. After possibly exchanging the names of V and W , there exist a primitive disk J in V and its dual disk R in W such that $\phi(J) = J$, and R is disjoint from $\phi(R)$.*

Proof. The involution ϕ of L defines a simplicial involution, denoted by $\overline{\phi}$, of the primitive tree $\mathcal{PT}(V)$. Any finite-order automorphism of a tree has a fixed point, and hence there exists a point v of $\mathcal{PT}(V)$ fixed by the involution $\overline{\phi}$. The point v is either a vertex of $\mathcal{PT}(V)$ or the barycenter of an edge of $\mathcal{PT}(V)$.

Suppose first that v is a vertex of $\mathcal{PT}(V)$. Then v is either a vertex of $\mathcal{P}(V)$ or the barycenter of a 2-simplex of $\mathcal{P}(V)$. The latter case occurs only for the lens space $L(3, 1)$ (see the definition of $\mathcal{PT}(V)$), but in this case, $\overline{\phi}$ fixes at least one of the three vertices of the 2-simplex. Thus, in any cases, $\overline{\phi}$ fixes a vertex of $\mathcal{P}(V)$, the isotopy class of a primitive disk, say E , in V . Since ϕ is an involution of V , ∂E is isotopic to a loop invariant under ϕ (for we may assume that ϕ is an isometry with respect to a hyperbolic structure on ∂V , and then the unique geodesic in ∂V isotopic to ∂E is ϕ -invariant). Once ∂E is ϕ -invariant, we may replace E with a ϕ -invariant disk J by Lemma 4.1 (1). Since J is primitive, it has a dual disk D in W . We may assume that D and $\phi(D)$ meet transversely. By Lemma 4.1 (1) and (2), we may replace D with a dual disk R such that R and $\phi(R)$ are disjoint.

Next, suppose that v is the barycenter of an edge of $\mathcal{PT}(V)$, and then the involution $\overline{\phi}$ exchanges the two end vertices of the edge. We note that L cannot be $L(3, 1)$ in this case since a white vertex and a black vertex of $\mathcal{PT}(V)$ cannot be exchanged by the action of \mathcal{G} . Thus, the edge is one of the edges of $\mathcal{P}(V)$ (the edges of type-0, type-1 and type-2) or bridges.

Suppose that the edge is an edge of type-0 or a bridge. Then we have $\phi_*(x) = qx$ by Lemma 3.2 (2), but the involution ϕ satisfies $\phi_*(x) = -x$, which implies $q = 1$. This is a contradiction since $\mathcal{PT}(V)$ for $L(p, 1)$ contains neither an edge of type-0 nor a bridge. Therefore, the edge is of type-1 or type-2.

If the edge is of type-1, then the primitive pair of the end vertices of the edge have the unique common dual disk up to isotopy. We denote by v' the vertex of $\mathcal{P}(W)$ of the unique common dual disk, which is also fixed by $\bar{\phi}$. So we can go back to the first case by replacing V with W .

If the edge is of type-2, that is, L is $L(2, 1)$, then the primitive pair of the end vertices of the edge have exactly two common dual disks up to isotopy, which form a primitive pair in W . By Lemma 3.2 (1), the involution $\bar{\phi}$ also fixes each of the vertices of $\mathcal{P}(W)$ of the common dual disks. We denote by v' the vertex of one of the two common dual disks. So choosing the vertex v' of $\mathcal{P}(W)$ and replacing V with W , we can go back to the first case. \square

Proof of Theorem 1.1. It suffices to find meridian disks J_0 of V_0 and R_0 of W_0 such that ∂J_0 intersects ∂R_0 in a single point, J_0 intersects $K \cap V_0$ in a single point in its interior, and R_0 is disjoint from $K \cap W_0$. Then letting B_1 be the union of W_0 with a regular neighborhood of J_0 in V_0 , and $B_2 = \overline{S^3 \setminus B_1}$, we have a 2-bridge position for K that stabilizes to the original $(1, 1)$ -position. Let J and R be the primitive disks of V and W respectively obtained in Lemma 4.2. Since J is ϕ -invariant, we have either J intersects the arc $\hat{K} \cap V$ in a single point or J contains the arc $\hat{K} \cap V$. See Figure 4.

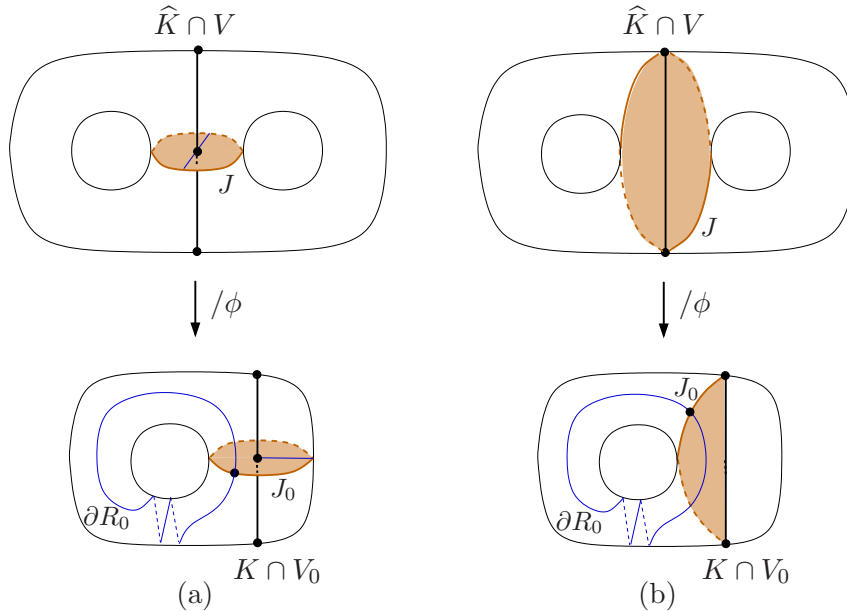


FIGURE 4. Two possibilities of the 2-fold cover V of V_0 branched along $K \cap V$.

First, consider the case that J intersects the arc $\hat{K} \cap V$ in a single point. Then the image J_0 of J is a meridian disk of V_0 that intersects $K \cap V_0$ in a single point in its interior. On the other hand the image R_0 of R is a disk properly embedded in W_0 since

R is disjoint from $\phi(R)$. Further R_0 is disjoint from $K \cap W_0$ since R is disjoint from $\widehat{K} \cap W$. In fact, R_0 should be an essential disk since ∂R_0 intersects ∂J_0 in a single point. That is, R_0 is the desired meridian disk and so we are done.

Next, suppose that J contains $\widehat{K} \cap V$. Then the image J_0 is a bigon in V_0 bounded by the arc $K \cap V_0$ and the arc c in ∂V_0 that is the image of ∂J_0 . See Figure 4 (b). As in the first case, the image R_0 of R is a disk properly embedded in W_0 that is disjoint from $K \cap W_0$. Further, since the arc c intersects ∂R_0 once and transversely in its interior, R_0 is non-separating in W_0 . In fact, $K \cap W_0$ should intersect R_0 otherwise. Consequently, R_0 is a meridian disk of W_0 . Now we move the arc $K \cap W_0$ to an arc d by isotopy so that (1) d lies in $\partial W_0 \setminus \partial R$, and (2) the arcs c and d meet only in their end points. Then K is isotopic to the knot $K_0 = c \cup d$ which lies in ∂V_0 and intersects ∂R_0 once and transversely. Then K_0 is a $(k, 1)$ -torus knot for some k , a trivial knot, that is a contradiction. \square

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